OFFICE OF NAVAL RESEARCH

Contract N7onr-35810

NR-360-003

Technical Report No. 20

IMPACT OF FINITE BEAMS OF DUCTILE METAL

by

P. S. Symonds and C.-F. A. Leth

GRADUATE DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R. I.

October, 1953

B11-20/22

IMPACT OF FINITE BEAMS OF DUCTILE METAL[1]

by

P. S. Symonds and C.-F. A. Leth

(Brown University)

Abstract

An analysis is here presented of the large plastic deformations of a beam under impact such as that due to a blow of a massive hammer, in which one cross-section is suddenly forced to move with a given velocity. The analysis treats both the case in which the velocity of the struck section is maintained constant until the total permanent deformation is acquired and cases of interrupted impacts in which the force at the struck section is suddenly removed after an arbitrary contact time. A complete solution in general non-dimensional form is obtained in a simple manner by basing the analysis on the assumption of "plastic-rigid" behavior, and results can be expected to be valid when the plastic deformations are large enough. Criteria for the validity of the present results are discussed, based on the major assumptions of the analysis.

---

1. The results in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.

1. Introduction

We consider the problem of a uniform beam of arbitrary length, initially at rest, which is subjected to a concentrated impact load at its mid-point such that the mid-section of the beam instantaneously acquires a velocity V which is then maintained constant. Bohnenblust [1]* treated the corresponding problem for the case of an infinitely long beam, in general terms for an arbitrary moment-curvature relation. It was recently discussed again by Conroy [2], who investigated the simplifications obtainable by neglecting the elastic part of the deformations, and took, in particular, moment-curvature relations of the two types shown in Fig. 1; again only infinitely long beams were considered.

The solution presented here is another example of the analysis of large plastic deformations in a finite structure subjected to a dynamic load, on the basis of the moment-curvature relation of Fig. 1(a). In this "plastic-rigid" type of analysis it is assumed that infinitely large curvatures can occur at cross-sections where the bending moment maintains the magnitude M_0, the "limit moment" or "fully plastic moment" used in limit analysis of structures under static loads. The concept of localization of deformations at such "plastic hinge" sections is assumed to be applicable to problems of dynamic loading of structures of ductile metal provided the energy absorbed in plastic deformations greatly exceeds that which could be absorbed in a wholly elastic manner. Based on this hypothesis criteria were given in a previous paper [3] for the validity of the solutions obtained, and similar criteria are given in the present paper for the new

* Numbers in square brackets refer to the Bibliography at the end of the paper.

solution presented here.

Earlier papers [3, 4, 5] dealt with finite beams subjected to force pulses with specified magnitudes and shape. Our main purpose in treating the present problem was to obtain a solution which could be treated more easily experimentally than the previous solutions obtained; the imposition of a known velocity involves fewer experimental difficulties than the application and measurement of a specified force-time curve. The present problem was emphasized by Vigness [6] in discussion of [3] as of interest from the point of view of experimental verification of the general method.

2. Analysis

For simplicity we treat the problem of a beam moving initially with velocity V normal to its length. At time $t = 0$ the mid-section is instantaneously brought to rest by contact with a rigid stop. It is obvious that the subsequent deformations will be identical with those which would be produced if the bar, initially at rest, were struck so that the mid-section suddenly acquires and maintains the velocity V. The beam is taken to have uniform mass per unit length m, limit moment M_0, and length 2ℓ. A typical cross-section will be specified by its distance x from the center of the beam, (Fig. 2).

For a sufficiently small time interval after the bar strikes the stop, its deformation will be indistinguishable from that of the infinite beam treated by Conroy [2]. Hence it will be expected that the force on the beam will vary initially as $1/\sqrt{t}$, and that plastic hinges will occur at the mid-section and

at two cross-sections each at a distance x_h from the center; x_h will be expected to increase initially in proportion to $\sqrt{t}$. The analysis of the present problem will therefore be based on the diagrams of Fig. 3. From the symmetry of the problem it is obviously enough to consider either half of the beam. The segments OH and HB are subjected to end moments as shown. At the center of the beam there is a reaction force R exerted by the stop so that a shear force R/2 acts as shown on the one segment. However, there is no shear force at cross-section $x = x_h$, where the travelling hinge is located at a given instant, since this is a cross-section where the bending moment has a relative maximum; thus $dM/dx = Q = 0$. The loading on the segments includes also the distributed inertia forces due to the accelerations. It can be verified that although they are changing in length, at any instant the two segments move as rigid bodies (hinged together as shown) since at any instant the complete set of loads is such that the bending moment reaches the magnitude M_o only at the sections $x = 0$ and $x = x_h$. The lateral hinge moves outward (i.e., x_h increases with time) so that the segment OH acquires a permanent deformation, but since the moment in the interior of the segment HB is always less than M_o in magnitude the segment to the right of the travelling hinge at any instant is undeformed.

Let the angular velocities of the segments OH and HB be ω_o and ω_1, respectively. Writing moment-angular acceleration equations with respect to the fixed point O in one case and the

center of gravity of the segment in the other, we obtain[1]

$$\frac{1}{3} m x_h^3 \frac{d\omega_o}{dt} = - 2M_o \qquad (1)$$

$$\frac{1}{12} m(\ell - x_h)^3 \frac{d\omega_1}{dt} = M_o. \qquad (2)$$

The following equation expresses the fact that the change of moment of momentum of the half-beam OB with respect to an axis at O is equal to the angular impulse of the moment M_o acting during the time t:

$$-M_o t = [\frac{1}{6} m(\omega_o - \omega_1)(3\ell^2 x_h - x_h^3) + \frac{1}{3} m\ell^3\omega_1] - \frac{1}{2} m\ell^2 V. \qquad (3)$$

The expression in brackets is the moment of momentum at any time t, while the last term is the initial moment of momentum of the half-beam.

Equations (1) - (3) are the basic equations of the analysis from which, together with appropriate initial conditions the three unknowns x_h, ω_o, ω_1 are to be found. It is now convenient to rewrite the equations in terms of new dimensionless variables defined as follows:

1. These equations are correct despite the fact that x_h is a function of time, because the velocities of elements just to the left and just to the right of the travelling hinge are equal at any instant. Thus, although the segment of length x_h is increasing in length, the element joining it in time dt comes in with the same velocity as the end of the segment, and there is no impulsive contribution to the momentum.

$$\Omega = \frac{\ell}{V}\,\omega_0; \qquad \psi = \frac{\ell}{V}\,\omega_1, \tag{4}$$

$$\eta = \frac{M_o t}{m\ell^2 V}; \qquad \xi = \frac{x_h}{\ell}\,. \tag{5}$$

Equations (1) - (3) now take the following forms:

$$\Omega' \equiv \frac{d\Omega}{d\eta} = -\,\frac{6}{\xi^3}\,, \tag{6a}$$

$$\psi' \equiv \frac{d\psi}{d\eta} = \frac{12}{(1-\xi)^3}\,, \tag{6b}$$

$$3 - 6\eta = (\Omega - \psi)(3\xi - \xi^3) + 2\psi. \tag{6c}$$

We will henceforth use a prime to denote differentiation with respect to η. The initial conditions may be taken as

$$\eta = 0; \qquad \xi = 0; \qquad \psi = 0. \tag{7}$$

From Eq. (6c) this implies that $(\Omega\xi)_o = 1$, so that in this type of impact there is initially a singularity in the angular velocity at the point of impact.

We begin the solution of Eqs. (6) by differentiating Eq. (6c) with respect to η, making use of Eqs. (6a) and (6b) and simplifying to obtain the relation

$$\Omega - \psi = \frac{6}{\xi'}\left[\frac{1}{\xi^2} - \frac{1}{(1-\xi)^2}\right]. \tag{8}$$

For completeness, we note that the above result may be derived from a quite different viewpoint, namely by differentiating the equation which expresses the fact (as shown in [3]) that velocities are continuous at the hinge section. Equation (8) can thus be identified as expressing the fact that accelerations are

discontinuous across the moving hinge section.

A second differentiation of Eq. (8), substitution again of Ω' and ψ' from Eqs. (6a) and (6b), and rearrangement leads to the following equation for ξ:

$$\left[\frac{1 - 2\xi}{(1 - \xi)^2}\right]\xi\xi'' + \xi'^2 = 0, \qquad (9)$$

which is to be solved subject to the initial conditions (7). A first integration of Eq. (9) can easily be performed since the independent variable does not appear explicitly. Let $\xi' = p(\xi)$; then

$$\xi'' = \frac{dp(\xi)}{d\xi}\frac{d\xi}{d\eta} = \xi'\frac{dp}{d\xi} = p\frac{dp}{d\xi}.$$

With this substitution Eq. (9) becomes

$$\left[\frac{1 - 2\xi}{(1 - \xi)^2}\right]\xi\frac{dp}{d\xi} + p = 0$$

if it is assumed that $p \neq 0$. Integration of the above yields

$$p = \frac{d\xi}{d\eta} = \frac{1}{A}\frac{e^{\xi/2}(1 - 2\xi)^{1/4}}{\xi}.$$

Thus the general solution of Eq. (9) can be written in the form

$$\eta = A\int_0^{\xi}\frac{ze^{-z/2}\,dz}{(1 - 2z)^{1/4}} + B \qquad (10)$$

where A and B are constants to be evaluated by means of the initial conditions Eq. (7). The numerical evaluation of the integral in Eq. (10) is speeded by making the substitution $(1 - 2z) = 4s$. Integration by parts and rearrangement then leads to the following form:

$$\eta = A[e^{-\xi/2}(1 - 2\xi)^{3/4} + 2F(\xi)] + B \qquad (11)$$

where

$$F(\xi) = \sqrt{2}\, e^{-1/4} \int_{\frac{1}{4}(1-2\xi)}^{1/4} s^{-1/4} e^{s} ds - \frac{1}{2} .$$

The advantage of this form appears when the integrand in Eq. (11) is expanded in a power series; term by term integration then leads to the following rapidly converging series for $F(\xi)$:

$$F(\xi) = 0.07952 - 2e^{-1/4}(1 - 2\xi)^{3/4} \sum_{n=0}^{\infty} \frac{(1 - 2\xi)^n}{n!(3 + 4n)4^n}.$$

Equations (6) and (8) can now be integrated to give the dimensionless angular velocities as follows:

$$\Omega = \frac{3}{2} + 6A\, [\frac{1}{\xi}\, (1 - 2\xi)^{3/4}\, e^{-\xi/2} - F] - 3B \qquad (12a)$$

$$\psi = \frac{3}{2} - 6A\, [\frac{(2 - \xi)(1 - 2\xi)^{3/4}\, e^{-\xi/2}}{(1 - \xi)^2} + F] - 3B. \qquad (12b)$$

After applying the initial conditions (7) to Eqs. (10) and (12b) we find that A and B must have values $B = 0$, $A = 1/6$. Thus ξ and η are related by

$$6\eta = e^{-\xi/2}\, (1 - 2\xi)^{3/4} + 2\sqrt{2}\, e^{-1/4} \int_{\frac{1}{4}(1-2\xi)}^{1/4} s^{-1/4} e^{s} ds - 1 \quad (13)$$

It can be seen from the denominator in Eq. (10) that the above solution is valid only for $0 \leq \xi < \frac{1}{2}$. This is the range we are interested in, as will be seen shortly.

3. Numerical results

The deformations of main interest are the angles Θ_0 and Θ_1 at mid-point and tip, respectively, and the permanent curvature $\varkappa$ of the part of the bar through which the lateral hinge has travelled. We have the following general formulas for the angles

$$\Theta_0 = \int_0^t \omega_0 dt = \frac{m\ell V^2}{M_0} \int_0^\xi \frac{\Omega}{\xi'} d\xi \tag{14}$$

$$\Theta_1 = \int_0^t \omega_1 dt = \frac{m\ell V^2}{M_0} \int_0^\xi \frac{\psi}{\xi'} d\xi \, . \tag{15}$$

The change $d\Theta$ of the angle of the beam across the lateral hinge has the value

$$d\Theta = (\omega_0 - \omega_1)dt.$$

Then

$$\varkappa = \frac{d\theta}{dx} = (\omega_0 - \omega_1) \frac{dt}{dx} = \frac{mV^2}{M_0} \frac{\Omega - \psi}{\xi'}$$

$$\varkappa = \frac{mV^2}{6M_0} \frac{(1 - 2\xi)^{1/2} e^{-\xi}}{(1 - \xi)^2} \, . \tag{16}$$

Numerical results are tabulated in Table I.

Table I

ξ	η	Ω	ψ	$\frac{M_0}{m\ell V^2}\Theta_0$	$\frac{M_0}{m\ell V^2}\Theta_1$	$\frac{M_0}{mV^2}\varkappa$
0	0	∞	0	0	0	0.1667
0.05	0.0002	19.524	0.003	0.0083	0.0000	0.1667
0.10	0.0008	9.045	0.012	0.0167	0.0000	0.1665
0.15	0.0019	6.583	0.031	0.0250	0.0000	0.1661
0.20	0.0034	4.883	0.063	0.0334	0.0001	0.1652
0.25	0.0053	3.845	0.114	0.0418	0.0003	0.1632
0.30	0.0078	3.136	0.191	0.0502	0.0007	0.1594
0.35	0.0108	2.610	0.309	0.0588	0.0013	0.1523
0.40	0.0144	2.191	0.491	0.0676	0.0029	0.1388
0.45	0.0191	1.829	0.786	0.0769	0.0059	0.1111
0.50	0.0265	1.421	1.421	0.0888	0.0138	0.0000

The previous analysis is based on the configuration of Fig. 3. This type of motion continues until the angular velocities of the inner and outer segments become equal, i.e., until $\Omega = \psi$. From Eq. (8) this happens when

$$\frac{1}{\xi_s^2} - \frac{1}{(1 - \xi_s)^2} = 0; \text{ therefore } \quad \xi_s = \frac{1}{2} \tag{17a}$$

where ξ_s denotes the final value of the hinge coordinate. The corresponding value of the dimensionless time is

$$\eta_s = 0.0265. \tag{17b}$$

For later times $\eta \geq \eta_s$ there is a plastic hinge at the mid-section only, and the two halves rotate as rigid bars pinned at one end. The equation of angular acceleration of the right-hand half, in dimensionless variables, is

$$\frac{d\Omega}{d\eta} = - 3. \tag{18}$$

Hence in this final phase of the motion we have

$$\Omega = - 3\eta + c_1$$

$$\frac{M_o}{m\ell V^2} \theta_o = - \frac{3}{2} \eta^2 + c_1\eta + c_2.$$

Making use of the conditions $\eta_s = 0.0265$, $\Omega_s = 1.421$, $\frac{M_o}{m\ell V^2} \theta_{os} = 0.0888$, as given in Table I, we obtain

$$\Omega = 1.500 - 3\eta \tag{19a}$$

$$\frac{M_o}{m\ell V^2} \theta_o = 0.0500 + 1.5\eta - 1.5\eta^2. \tag{19b}$$

These hold until $\Omega = 0$, or, from Eq. (19a), until $\eta = \eta_f$, where

$$\eta_f \equiv \frac{M_o t_f}{m\ell^2 V} = 0.500 \tag{20a}$$

The corresponding final value of the angle θ_o is

$$\theta_o = 0.425 \frac{m\ell V^2}{M_o} . \tag{20b}$$

The growth of the deformations with time is shown in Fig. 4.

Some simple checks on the above results are of interest. First, we note that the momentum relation Eq. (6c), evaluated at the time η_s, becomes

$$3 - 6\eta_s = 2\psi_s.$$

Taking values from the last line of Table I the two sides of the equation have values

$$3 - 0.1590 = 2.841; \quad 2 \times 1.421 = 2.842$$

which is a satisfactory check. Again, from Eq. (6c) the time η_f at which the deformation is completed is given by

$$3 - 6\eta_f = 0; \quad \eta_f = \frac{1}{2}$$

which verifies the value of Eq. (20a) computed from the tabulated values.

Finally, the energy relations may be examined. Before the beam hits the rigid stop it has the kinetic energy E

$$E = \frac{1}{2}(m2\ell)V^2 = m\ell V^2.$$

When the motion has ceased, this energy has been spent in work at plastic hinges. The total energy absorbed in plastic deformations is W, where

$$W = 2M_o\theta_{of} + 2(\theta_{of} - \theta_{1f})M_o.$$

Inserting values from Eq. (20b) and Table I this becomes

$$W = 2M_o(0.425\,\frac{m\ell V^2}{M_o}) + 2M_o(0.0750\,\frac{m\ell V^2}{M_o}) = 1.000\,m\ell V^2.$$

Thus the necessary energy balance is checked.

We may also compare the results obtained here with those obtained by Conroy [2]. It is there shown by equations (2) and (20) that $12\eta = \xi^2$ for an infinitely long beam. Our result should converge to this value for small ξ. We expand the right-hand side of Eq. (13) as a power series in ξ, as follows:

$$e^{-\xi/2}(1 - 2\xi)^{3/4} = 1 - 2\xi + \frac{1}{2}\,\xi^2 + 0(\xi^3)$$

$$\int_{\frac{1}{4}(1-2\xi)}^{1/4} \frac{e^s ds}{s^{1/4}} = \frac{\sqrt{2}}{2}\,e^{1/4}\,\xi + 0(\xi^3).$$

Hence Eq. (13) reduces, for small ξ, to

$$\eta = \frac{1}{12}\,\xi^2 + 0(\xi^3). \tag{21}$$

Furthermore Conroy shows by Eqs. (20) and (26) that the shear force at the center varies as $\frac{1}{2}\sqrt{\frac{3}{\eta}}\,\frac{M_o}{\ell}$. The moment equilibrium for the inner part of the beam around its center of gravity requires

$$\frac{1}{4}\,Rx_h - 2M_o = \frac{mx_h^3}{12}\,\dot{\omega}_o.$$

Transforming this into dimensionless coordinates we get

$$\mu\xi = 8 + \frac{1}{3}\,\xi^3\,\Omega' \tag{22}$$

where $\mu = \frac{R\ell}{M_o}$. With the use of equation (6a) we get

$$\mu = \frac{6}{\xi} . \qquad (23)$$

In the limit for small ξ we have $\xi = \sqrt{12\eta}$. Hence in the limit $\mu = \sqrt{\frac{3}{\eta}}$, which agrees with the result in [2].

For completeness we finish the analysis with the calculation for the motion of the beam if the support is removed at a time $t = \tau < \frac{m\ell^2 V}{M_o} \eta_f$. The subsequent motion ($t > \tau$) is to be determined by analysis appropriate to a beam acted on by no external loads. Such an analysis is described in [4]. There it is shown that when lateral hinges are present at the instant the central force is reduced to zero the lateral hinges then move with constant velocity until the angular velocities become equalized. By simple calculations, using Eqs. (20) - (29) of Ref [4], one can find the increments in angular velocities and displacements which occur after removal of the force at arbitrary times τ, corresponding to dimensionless times $\eta_\tau = \frac{M_o \tau}{m\ell^2 V} < 0.5$. Fig. 6 shows a curve of final deformation angles resulting from "interrupted impacts" of various durations, plotting $M_o \theta_{of}/m\ell V^2$ against $M_o \tau / m\ell^2 V$. The shape of the force-time curve concerned here is indicated in Fig. 5.

4. Criteria for Validity of Results

We consider now the implications for experimental comparisons of two major assumptions involved, namely the neglect of all elastic deformations by comparison with plastic deformations, and the neglect of shape changes throughout the analysis.

The first assumption can be expected to be valid [3] if the total energy absorbed in plastic deformations greatly exceeds the maximum possible amount of elastic strain energy that could

be stored in the beam. In the present case this implies that

$$2M_o\theta_{of} = m\ell V^2 >> \frac{M_o^2\ell}{EI}$$

where EI is the elastic flexural rigidity. Thus for the present results to be good approximations one requirement is that

$$V^2 >> \frac{M_o^2}{mEI} . \qquad (24a)$$

Alternatively the above requirement may be expressed as

$$V^2 \geq n \frac{M_o^2}{mEI} \qquad (24b)$$

where n is a number which presumably is of the order of 10.

A further requirement is set by the assumption of negligible shape changes, i.e., the use of coordinates referring to the undeformed rather than to the actual beam. This assumption actually restricts the magnitude of the central angle θ_o attained only during that part of the response in which deformations are occurring at lateral hinges; when only the central hinge is present the equations apply to deformation angles of unrestricted magnitude. From Table I the maximum value of θ_o obtained while the lateral hinges are present is $0.0888\ m\ell V^2/M_o$; if we assume that this does not exceed about 0.15 radians, we have the additional inequality:

$$0.0888 \frac{m\ell V^2}{M_o} < 0.15. \qquad (25)$$

This imposes a limitation on the length-depth ratio of the beam, if the requirement of inequalities (24) is taken into account. Let (24b) be taken as an <u>equation</u> and conbined with the inequality (25). We obtain

$$n \frac{\ell M_o}{EI} < \frac{0.15}{0.0888} \cong 1.7. \tag{26a}$$

But the limit moment M_o can be computed [7] as

$$M_o = \alpha\sigma_y Z = \alpha\sigma_y\left(\frac{2I}{h}\right) \tag{27}$$

where the "shape factor" α is a number of the order of unity which depends on the shape of the cross-section and has the value 1.5 for a rectangular cross-section; σ_y is the yield stress; and $Z = 2I/h$ is the elastic section modulus, i.e., quotient of moment of inertia I and half-depth h/2. Using this formula for M_o the inequality (26a) can be written as

$$\frac{h}{2\ell} > \frac{1}{1.7} n\alpha \frac{\sigma_y}{E} . \tag{26b}$$

Since σ_y/E is of the order of 10^{-3} for steel, and α will probably be about 1.5, it is seen that the restriction of small shape changes requires only that $h/2\ell$ exceed about 1/100, for n = 10. Hence it is clear that this restriction would introduce no experimental difficulties.

Finally the order of magnitude of V demanded by inequality (24b) is of interest. We use Eq. (27) and write also $m = \rho A$, $I = Ai^2$, $E/\rho = c_o^2$, and obtain

$$V \geq \sqrt{n}\, c_o\left(\frac{\sigma_y}{E}\right)\left(\frac{i\alpha}{h/2}\right) \tag{24c}$$

where

$c_o = \sqrt{E/\rho}$ = speed of longitudinal elastic waves

ρ = mass density

A = cross-sectional area

i = radius of gyration

h = depth of beam

α = shape factor (defined in connection with Eq. (27)).

Thus for steel with $c_o = 16{,}000$ ft/sec., $\sigma_y/E = 10^{-3}$ and a rectangular cross-section, if $n = 10$ we find that V must exceed about 45 ft/sec.

The main purpose of a laboratory test program would be to determine under what circumstances, if any, the plastic-rigid type of analysis yields satisfactory results for the major plastic deformations. The fundamental assumption is that elastic deformations are negligible. To find the range of usefulness of this assumption, a series of tests could be made with the objective of determining the value of the number n used in Eq. 24b, above which the rigid-plastic analysis predicts deformation in good agreement with those observed in the tests. If the elastic deformations are the only important physical effect which is ignored then such a series of tests will yield a definite value of n, such that when the total energy absorbed is at least n times the maximum possible elastic energy the beam could carry, the present type of analysis will be suitable. A value of n determined for the present case of velocity impact would probably have significance for other types of problems of dynamic loading.

In any experiments other physical effects will occur which have been neglected here, and these might turn out to be so important that the present type of analysis does not yield accurate results even though the criterion based on elastic energy is satisfied. Among the physical phenomena which have been ignored are:

(a) <u>strain-hardening</u>, which will occur to some degree at cross-sections where plastic hinge action occurs;

(b) speed of loading effect on the yield-stress and post-yield properties, as, for example, reported by Manjoine [8] and Nadai and Manjoine [9];

(c) effect of transverse shear forces which are known to reduce the limit moment below that which can be reached in pure bending; in the present problem the shear force at the struck section has large values in the initial instants of the impact;

(d) finite contact area of the hammer or stop; the study in Ref. [5] of distributed as compared with concentrated loads showed that even a very small degree of spreading of the load over a finite segment may appreciably reduce the magnitude of the final deformations;

(e) finite time of acceleration; it was assumed that the hammer or stop was perfectly rigid and that the velocity of the struck section was instantaneously acquired or annihilated; since any physical hammer or stop has finite rigidity there is a finite time of acceleration; in fact the contact will probably be intermittent during the initial instants of the impact, as is known to be the case in the elastic impact of a mass on a beam.

All but one of the foregoing effects, if taken into account, would tend to reduce the deformations below those given by the present analysis. The single exception is the effect of transverse shear forces, which tend to weaken the beam and hence if properly taken into account would cause the analysis to predict larger deformation magnitudes than those of the present theory.

If the transverse shear effect is minimized by use of compact sections and fairly large ratios of span to depth, the other effects would presumably predominate, and cause the present analysis to overestimate the deformations.

Finally, it should be re-emphasized that a fundamental presupposition throughout the paper is that the material has sufficient ductility under dynamic conditions so that rupture does not occur before the predicted final deformations are attained. The present type of analysis should be regarded as a basic one, particularly attractive for its simplicity and generality, but which may have to be refined in particular cases. Carefully planned and interpreted tests are needed. They will be an invaluable aid in assessing the range of usefulness of the present type of analysis and in showing the directions in which refinements are most urgent.

Bibliography

1. P. E. Duwez, D. S. Clark and H. F. Bohnenblust, "The Behavior of Long Beams Under Impact Loading", Journal of Applied Mechanics, Trans. ASME, vol. 72, 1950, pp. 27-34.

2. M. F. Conroy, "Plastic-Rigid Analysis of Long Beams Under Transverse Impact Loading", Journal of Applied Mechanics, vol. 19, No. 4, 1952, pp. 465-471.

3. E. H. Lee and P. S. Symonds, "Large Plastic Deformations of Beams under Transverse Impact", Journal of Applied Mechanics, vol. 19, No. 3, 1952, pp. 308-315.

4. P. S. Symonds, "Dynamic Load Characteristics in Plastic Bending of Beams", Paper No. 53-APM-26 to be presented at National Conference of the Applied Mechanics Division, Minneapolis, Minn., June 18-20, 1953; to be published in Journal of Applied Mechanics.

5. J. A. Seiler and P. S. Symonds, "Plastic Deformations in Beams under Symmetric Dynamic Loading", Technical Report No. 13 of Brown University to Office of Naval Research, April. 1953.

6. I. Vigness, Discussion of Reference 3, Journal of Applied Mechanics, vol. 20, No. 1, 1953, pp. 151-152.

7. P. S. Symonds and B. G. Neal, "Recent Progress in the Plastic Methods of Structural Analysis", Journal of The Franklin Institute, vol. 252, 1951, pp. 383-407, 469-492.

8. M. J. Manjoine, "Influence of Rate of Strain and Temperature on Yield Stresses of Mild Steel", Jour. Applied Mechanics vol. 11, pp. 211-218, 1944.

9. A. Nadai and M. J. Manjoine, "High-Speed Tension Tests at Elevated Temperatures - Parts II & III". Jour. Applied Mechanics vol. 8, p. 77, 1941.

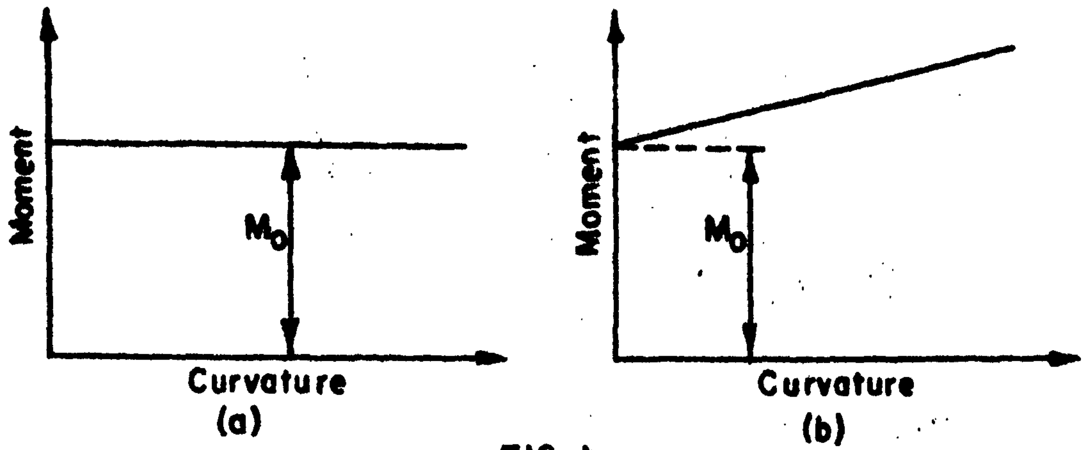


FIG. 1

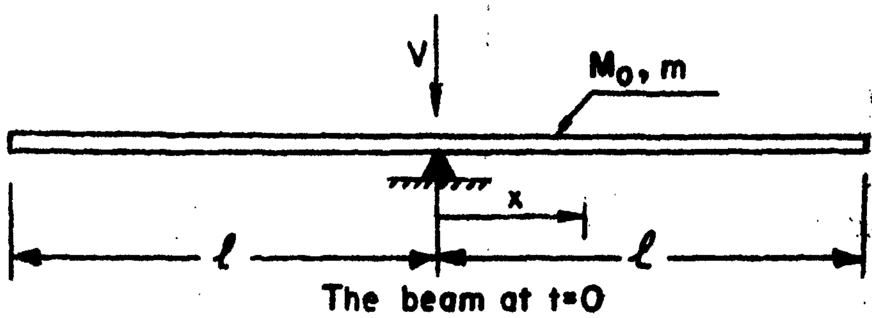


FIG. 2

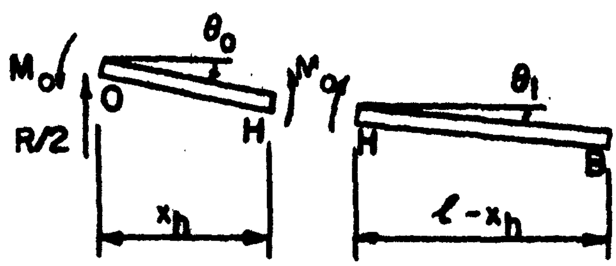


FIG. 3

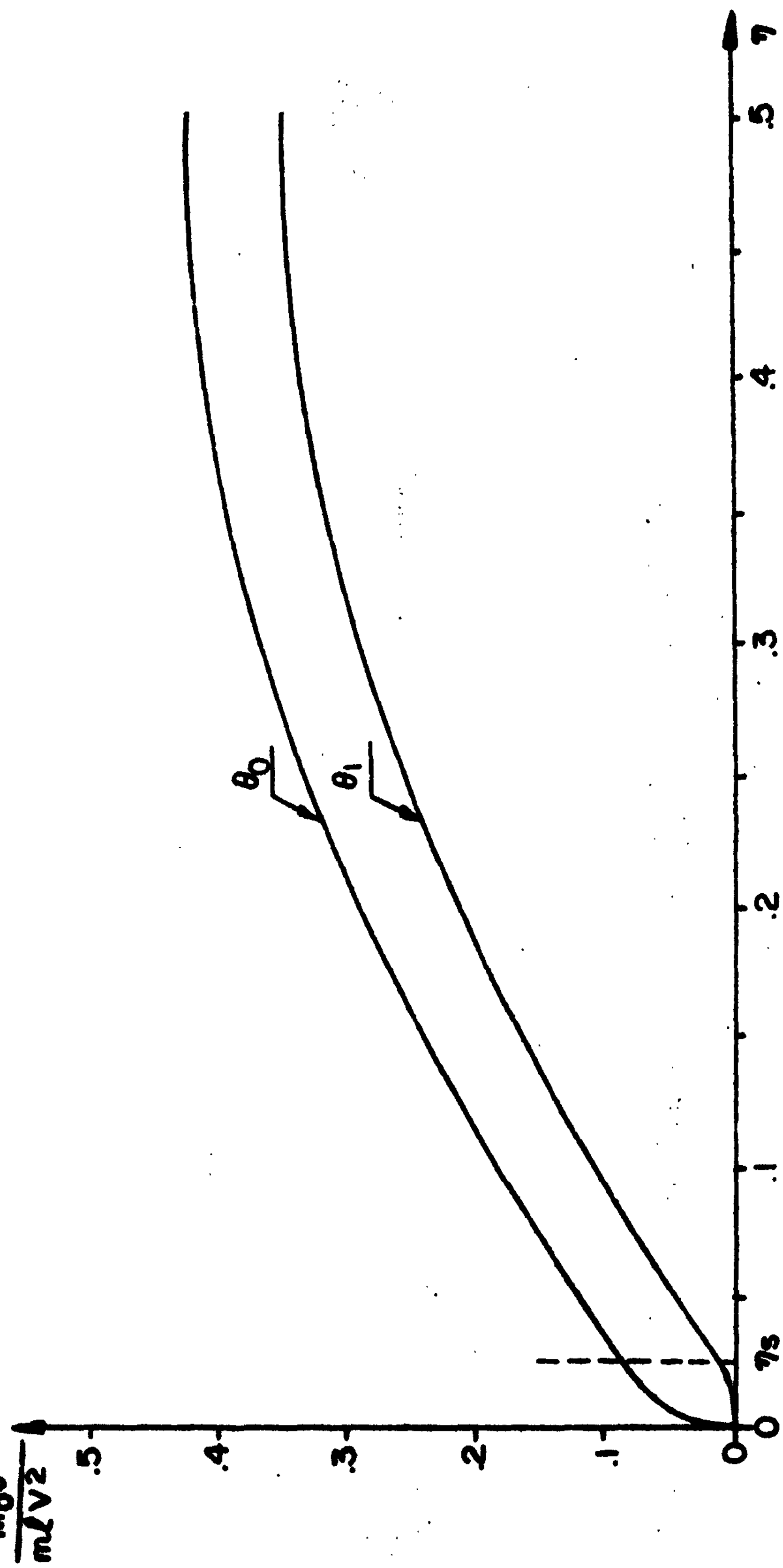
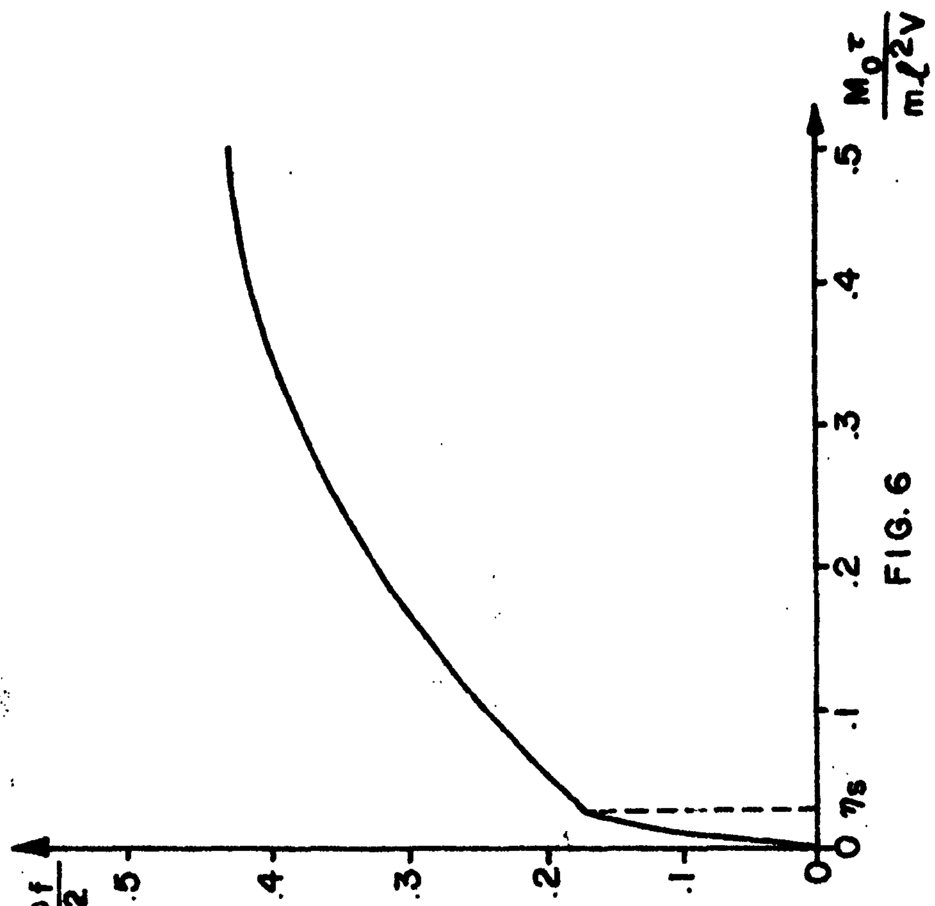
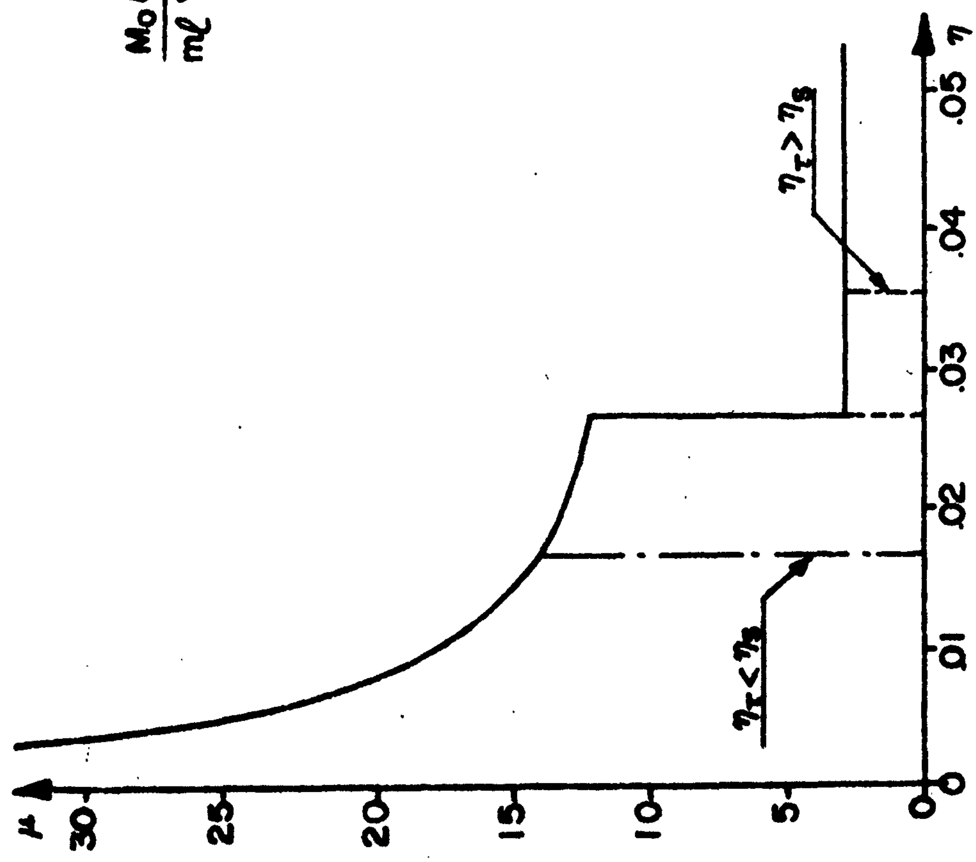


FIG. 4



$\frac{M_o \theta_{of}}{m\ell V^2}$
.5
.4
.3
.2
.1
0
η_s
.1
.2
.3
.4
.5
$\frac{M_o \tau}{m\ell^2 V}$

FIG. 6



μ
30
25
20
15
10
5
0
.01
.02
.03
.04
.05
η
$\eta_\tau < \eta_s$
$\eta_\tau > \eta_s$

FIG. 5

Distribution List
for
Technical and Final Reports Issued Under
Office of Naval Research Project NR-360-364, Contract N7onr-35810

I: Administrative, Reference and Liaison Activities of ONR

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 438 (2)
Code 432 (1)
Code 466(via Code 108)(1)

Director, Naval Research Lab.
Washington 25, D. C.
Attn: Tech. Info. Officer (9)
Technical Library (1)
Mechancics Division (2)

Commanding Officer
Office of Naval Research
Branch Office
495 Summer Street
Boston 10, Mass. (2)

Commanding Officer
Office of Naval Research
Branch Office
346 Broadway
New York 13, New York (1)

Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco, California (1)

Commanding Officer
Office of Naval Research
Branch Office
1030 Green Street
Pasadena, California (1)

Officer in Charge
Office of Naval Research
Branch Office, London
Navy No. 100
FPO, New York, N.Y. (5)

Library of Congress
Washington 25, D. C. (2)
Attn: Navy Research Section

Commanding Officer
Office of Naval Research
Branch Office
844 N. Rush Street
Chicago 11, Illinois (1)

II: Department of Defense and other interested Gov't. Activities

a) General

Research & Development Board
Department of Defense
Pentagon Building
Washington 25, D. C.
Attn: Library(Code 3D-1075) (1)

Armed Forces Special Weapons
Project
P.O. Box 2610
Washington, D. C.
Attn: LtCol. G.F. Blunda (2)

Joint Task Force 3
12St. & Const. Ave., N.W.
(Temp. U)
Washington 25, D.C.
Attn: Major B.D. Jones (1)

b) Army

Chief of Staff
Department of the Army
Research & Development Div.
Washington 25, D. C.
Attn: Chief of Res.&Dev. (1)

Office of the Chief of Engineers
Assistant Chief for Works
Department of the Army
Bldg. T-7, Gravelly Point
Washington 25, D.C.
Attn: Structural Branch
(R.L. Bloor) (1)

Engineering Research and
Development Laboratory
Fort Belvoir, Virginia
Attn: Structures Branch (1)

Army (cont.)

Office of the Chief of Engineers
Asst. Chief for Military
Construction
Department of the Army
Bldg. T-3, Gravelly Point
Washington 25, D. C.
Attn: Structures Branch
(M. F. Carey) (1)
Protective Construction
Branch (I. O. Thornley) (1)

Office of the Chief of Engineers
Asst. Chief for Military
Operations
Department of the Army
Bldg. T-7, Gravelly Point
Washington 25, D. C.
Attn: Structures Development
Branch (W.F. Woollard) (1)

U.S. Army Waterways Experiment
Station
P. O. Box 631
Halls Ferry Road
Vicksburg, Mississippi
Attn: Col. H. J. Skidmore (1)

The Commanding General
Sandia Base, P. O. Box 5100
Albuquerque, New Mexico
Attn: Col. Canterbury (1)

Operations Research Officer
Department of the Army
Ft. Lesley J. McNair
Washington 25, D. C.
Attn. Howard Brackney (1)

Office of Chief of Ordnance
Office of Ordnance Research
Department of the Army
The Pentagon Annex #2
Washington 25, D. C.
Attn: ORDTB-PS (1)

Ballistics Research Laboratory
Aberdeen Proving Ground
Aberdeen, Maryland
Attn: Dr. C. W. Lampson (1)

c) Navy

Chief of Naval Operations
Department of the Navy
Washington 25, D. C.
Attn: OP-31 (1)
OP-363 (1)

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Director of Research (2)
Code 423 (1)
Code 442 (1)
Code 421 (1)

Director, David Taylor Model Basin
Department of the Navy
Washington 7, D. C.
Attn: Code 720, Structures
Division (1)
Code 740, Hi-Speed
Dynamics Div. (1)

Commanding Officer
Underwater Explosions Research Div.
Code 290
Norfolk Naval Shipyard
Portsmouth, Virginia (1)

Commander
Portsmouth Naval Shipyard
Portsmouth, N. H.
Attn: Design Division (1)

Director, Materials Laboratory
New York Naval Shipyard
Brooklyn 1, New York (1)

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Attn: Ad-3, Technical Library (1)
Rec, P. H. Girouard (1)

Naval Ordnance Laboratory
White Oak, Maryland
RFD 1, Silver Spring, Maryland
Attn: Mechanics Division (1)
Explosive Division (1)
Mech. Evaluation Div. (1)

Commander
U.S. Naval Ordnance Test Station
Inyokern, California
Post Office - China Lake, Calif.
Attn: Scientific Officer (1)

Naval Ordnance Test Station
Underwater Ordnance Division
Pasadena, California
Attn: Structures Division (1)

Navy (cont.)

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D.C.
Attn: TD-41, Technical Library (1)

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Code P-314 (1)
Code C-313 (1)

Officer in Charge
Naval Civil Engr. Research &
Evaluation Laboratory
Naval Station
Port Hueneme, California (1)

Superintendent
U.S. Naval Post Graduate School
Annapolis, Maryland (1)

d) Air Forces

Commanding General
U.S. Air Force
The Pentagon
Washington 25, D. C.
Attn: Res.& Development Div.(1)

Deputy Chief of Staff, Operations
Air Targets Division
Headquarters, U.S. Air Force
Washington 25, D. C.
Attn: AFOIN-T/PV (1)

Office of Air Research
Wright-Patterson Air Force Base
Dayton, Ohio
Attn: Chief, Applied Mechanics Group (1)

e) Other Government Agencies

U.S. Atomic Energy Commission
Division of Research
Washington, D. C. (1)

Director, National Bureau of Standards
Washington 25, D. C.
Attn: Dr. W.H. Ramberg (1)

Supplementary Distribution List

Addressee	No. of Copies	
	Unclassified Reports	Classified Reports
Professor Lynn Beedle Fritz Engineering Laboratory Lehigh University Bethlehem, Pennsylvania	1	-
Professor R.L. Bisplinghoff Dept. of Aeronautical Engineering Massachusetts Institute of Technology Cambridge 39, Massachusetts	1	1
Professor Hans Bleich Dept. of Civil Engineering Columbia University Broadway at 117th St. New York 27, New York	1	1

Addressee	Unclassified Reports	Classified Reports
Professor B.A. Boley Dept. of Aeronautical Engineering Ohio State University Columbus, Ohio	1	-
Professor G.F. Carrier 309 Pierce Hall Harvard University Cambridge, Massachusetts	1	1
Professor R.J. Dolan Dept. of Theoretical & Applied Mechanics University of Illinois Urbana, Illinois	1	-
Professor Lloyd Donnell Department of Mechanics Illinois Institute of Technology Technology Center Chicago 16, Illinois	1	-
Professor A.C. Eringen Illinois Institute of Technology Department of Mechanics Technology Center Chicago 16, Illinois	1	-
Professor B. Fried Dept. of Mechanical Engineering Washington State College Pullman, Washington	1	-
Mr. Martin Goland Midwest Research Institute 4049 Pennsylvania Avenue Kansas City 2, Missouri	1	-
Dr. J.N. Goodier School of Engineering Stanford University Stanford, California	1	-
Professor R.M. Hermes College of Engineering University of Santa Clara Santa Clara, California	1	1
Professor R.J. Hansen Dept. of Civil & Sanitary Engineering Massachusetts Institute of Technology Cambridge 39, Massachusetts	1	1

Addressee	Unclassified Reports	Classified Reports
Professor M. Hetenyi Walter P. Murphy Professor Northwestern University Evanston, Illinois	1	-
Dr. N.J. Hoff, Head Department of Aeronautical Engineering & Applied Mechanics Polytechnic Institute of Brooklyn Brooklyn 2, New York	1	1
Dr. J.H. Hollomon General Electric Research Laboratories 1 River Road Schenectady, New York	1	-
Dr. W.H. Hoppmann Department of Applied Mechanics Johns Hopkins University Baltimore, Maryland	1	1
Professor L.S. Jacobsen Department of Mechanical Engineering Stanford University Stanford, California	1	1
Professor J. Kempner Department of Aeronautical Engineering and Applied Mechanics Polytechnic Institute of Brooklyn 99 Livingston Street Brooklyn 2, New York	1	1
Professor George Lee Department of Aeronautical Engineering Renssalaer Polytechnic Institute Troy, New York	1	-
Professor Paul Lieber Department of Aeronautical Engineering Renssalaer Polytechnic Institute Troy, New York	1	1
Professor Glen Murphy, Head Department of Theoretical & Applied Mechanics Iowa State College Ames, Iowa	1	-
Professor N.M. Newmark Department of Civil Engineering University of Illinois Urbana, Illinois	1	1

Addressee	Unclassified Reports	Classified Reports
Professor Jesse Ormondroyd University of Michigan Ann Arbor, Michigan	1	-
Dr. W. Osgood Armour Research Institute Technology Center Chicago, Illinois	1	-
Dr. R.P. Petersen, Director Applied Physics Division Sandia Laboratory Albuquerque, New Mexico	1	1
Dr. A. Phillips School of Engineering Stanford University Stanford, California	1	-
Dr. W. Prager Graduate Division of Applied Mathematics Brown University Providence 12, R. I.	1	1
Dr. S. Raynor Armour Research Foundation Illinois Institute of Technology Chicago, Illinois	1	-
Professor E. Reissner Department of Mathematics Massachusetts Institute of Technology Cambridge 39, Massachusetts	1	-
Professor M.A. Sadowsky Illinois Institute of Technology Technology Center Chicago 16, Illinois	1	-
Professor V.L. Salerno Department of Aeronautical Engineering Renssalaer Polytechnic Institute Troy, New York	1	1
Professor M.G. Salvadori Department of Civil Engineering Columbia University Broadway at 117th Street New York 27, New York	1	-
Professor J.E. Stallmeyer Talbot Laboratory Department of Civil Engineering University of Illinois Urbana, Illinois	1	1

Distribution List

Addressee	Unclassified Reports	Classified Reports
Professor E. Sternberg Illinois Institute of Technology Technology Center Chicago 16, Illinois	1	-
Professor R. G. Sturm Purdue University Lafayette, Indiana	1	-
Professor F. K. Teichmann Department of Aeronautical Engineering New York University University Heights, Bronx New York, N. Y.	1	-
Professor C. T. Wang Department of Aeronautical Engineering New York University University Heights, Bronx New York, N. Y.	1	-
Project File	2	2
Project Staff	5	-
For possible future distribution by the University	10	-
To ONR Code 438, for possible future distribution	-	10